Math 4550 - Homework # 2 - Subgroups

Part 1 - Computations

1. (a) Compute the order of every element of $\mathbb{Z}_4$.
   (b) Do the same for $\mathbb{Z}_5$.
2. Compute the order of every element in U_6.
3. Compute the order of every element in D_6.
4. In $\mathbb{Z}_8$ calculate these cyclic subgroups: $\langle \overline{2} \rangle$ and $\langle \overline{4} \rangle$ and $\langle \overline{5} \rangle$.
5. Calculate the cyclic subgroup $\langle e^{2\pi i/4} \rangle$ in U_8.
   Draw a picture of U_8 and circle the subgroup.
6. Describe the elements of $\langle 3 \rangle$ in $\mathbb{R}^*$.
7. Consider the group $GL(2, \mathbb{R})$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
   (a) Show that S is in $GL(2, \mathbb{R})$.
   (b) Show that S has order 4 and list the elements of $\langle S \rangle$.
8. Consider $GL(2, \mathbb{R})$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Describe the elements of $\langle T \rangle$.

Part 2 - Proofs

9. Show that $H = \{1, s, sr, sr^2\}$ is not a subgroup of D_6.
10. Prove that $H = \{1, r^2, s, sr^2\}$ is a subgroup of D_8.
11. Show that
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$
is a subgroup of $GL(2, \mathbb{R})$
12. Show that
$$H = \{2x + 3y \mid x, y \in \mathbb{Z}\}$$
is a subgroup of $\mathbb{Z}$.
13. Let G be an abelian group. Let $H = \{x \in G \mid x^2 = e\}$. Prove that H is a subgroup of G.

14. Let G be a group. Let H and K be subgroups of G. Prove that $H \cap K$ is a subgroup of G.

15. Let G be an abelian group. Let H and K be subgroups of G. Prove that

$$HK = \{hk \mid h \in H \text{ and } k \in K\}$$

is a subgroup of G.

16. Let G be a group. The center of G is

$$Z(G) = \{x \in G \mid xy = yx \text{ for all } y \in G\}$$

Prove that $Z(G)$ is a subgroup of G.
